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HYDRODYNAMICS

CONDITIONS AT THE FRONT OF A TRANSLATION WAVE DISTURBING A STEADY MOTION OF A REAL FLUID

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Let us consider the motion of a continuous wave. The equations of a gradually varying unsteady motion of a real fluid in open channels are of the following form (1):

the continuity equation is

$$\frac{\partial Q}{\partial s}+\frac{\partial F}{\partial t}=0 \tag{1}$$

and the dynamics equation is

$$-\frac{1}{B}\frac{\partial F}{\partial s}=\frac{U^2}{C^2R}-i_0+\frac{\alpha}{g}U\frac{\partial U}{\partial s}+\frac{\alpha}{g}\frac{\partial U}{\partial t} \tag{2}$$

Making use of the continuity equation, we can rewrite the dynamic equation as follows (2):

$$\frac{\partial F}{\partial s}\left(U^2-\frac{\partial F}{\alpha B}\right)=\frac{\partial Q}{\partial t}-2U\frac{\partial F}{\partial t}+\frac{gF}{\alpha}\left(\frac{U^2}{C^2R}-i_0\right) \tag{3}$$

In the sequel a formula of the Manning type $C^2=R^3/n^2$ will be used for the coefficient C * and we shall adopt the notations $1-\frac{1}{3}\frac{F}{B}\frac{dB}{dF}=N$ and $1-R\frac{d\chi}{dF}=1-2\frac{R}{B}\sqrt{1+m^2}=M$, supposing, in order to simplify the calculation, that the channels have a prismatic form.

By introducing a function whose meaning is determined by the continuity equation, the equation (3) can be reduced to a partial differential equation of the second order, which is not linear in the general case. The trajectory of the front of translation wave disturbing a steady flow is a characteristic (3). Along the characteristic the partial derivative cannot in general be computed. It is known, however, that if the condition establishing a relation between the independent variables and the first derivatives, with respect to these variables, of the function sought-for (in the present case, between s, t, dt/ds, F, Q, dF/ds, dQ/ds) is fulfilled, then one of the partial derivatives remains arbitrary, and it is possible to compute the derivatives along the characteristic.

* The results will not be affected essentially, if another type of formula instead of the Manning type is used, or, in general, if the term U^2/C^2R is replaced by a different form of resistance law, bringing on different relations (for instance, that of Bazin) or a different power of U in the equations (2) and (3).

Let us show that this condition is fulfilled at the front of a wave disturbing a steady flow. A convenient way of computing the derivatives will also be indicated, and the formulae will be derived for computing the derivatives of the first and second order in equation (3) for any distance of the front from the initial section.

In the initial section ($s=0$) either the hydrograph $Q=Q_i(t)$ or the water-gauge graph $H=H_i(t)$ is known, *i. e.* $F=F_i(t)$.

The fundamental condition at the wave-front, from which follow all the other conditions and which takes place in virtue of the fact that the initial motion is a steady one (the discharge is a constant all along the flow), is as follows:

$$Q=Q_i=\text{const};\ dQ/dt=0;\ d^2Q/dt^2=0,\ etc. \tag{4}$$

The relation between s and t is established through the introduction of the propagation velocity W of a point of the wave-front along the free surface of the steady flow. Then

$$s=\int_0^t W dt \tag{5}$$

and from (4) and (1) we have

$$\frac{\partial Q}{\partial t}=-W\frac{\partial Q}{\partial s}=W\frac{\partial F}{\partial t} \tag{6}$$

$$\frac{\partial^2 Q}{\partial t^2}=2W\frac{\partial^2 F}{\partial t^2}+W^2\frac{\partial^2 F}{\partial t\,\partial s}+\frac{dW}{dt}\frac{\partial F}{\partial s} \tag{7}$$

etc. The velocity of propagation of the wave-front coincides with the velocity ω_Q of constant discharge.

Let us find the relation between dF/dt and s, and from it establish the relation between F, $U=Q/F$, s, t and $ds/dt=W$. The velocity of the propagation of the wave-front is defined by a well-known formula [3]

$$W=U\pm\sqrt{\frac{gF}{\alpha B}} \tag{8}$$

In the case of a uniform motion ($dF/dt=0$) the velocity of the front W coincides also with the velocity ω of propagation of constant area F (or constant depth H).

For a steady gradually varying non-uniform flow in prismatic channels the relation between s and F, or between s and U is defined by the equation of Belanger [4]

$$\frac{dF}{ds}=\frac{gF}{\alpha}\left(\frac{U^2}{C^2R}-i_0\right)\Big/\left(U^3-\frac{gF}{\alpha B}\right) \tag{9}$$

The velocity W can therefore be computed for any position of the front. Accordingly, the relations between s, t and ds/dt and also between F and t are known, too (the latter relation for $\beta=0$ can be computed from the tables given in [3]).

For the general case of non-uniform motion we have:

$$\frac{dF}{dt}=\frac{\partial F}{\partial s}W+\frac{\partial F}{\partial t} \tag{10}$$

Hence

$$\frac{\partial F}{\partial t}=W\left(\frac{dF}{ds}-\frac{\partial F}{\partial s}\right);\quad \frac{\partial Q}{\partial t}=W^2\left(\frac{dF}{ds}-\frac{\partial F}{\partial s}\right) \tag{11}$$

For the derivative which remains arbitrary we, respectively, choose $\partial F/\partial s=r$, $\partial^2 F/\partial s^2=\delta$, *etc*. These derivatives characterize the shape of the free surface (its inclination to the bed, its curvature, *etc*.). Construct the differential equation for the derivative $\partial F/\partial s=r$. We shall have

$$\frac{dr}{dt}=\frac{\partial^2 F}{\partial s^2}W+\frac{\partial^2 F}{\partial s\,\partial t} \tag{12}$$

$$\frac{\partial^2 F}{\partial t^2}=\frac{d^2 F}{dt^2}-2W\frac{dr}{dt}-\frac{dW}{dt}\frac{\partial F}{\partial s}+W^2\frac{\partial^2 F}{\partial s^2} \tag{13}$$

$$\frac{\partial^2 Q}{\partial t^2}=2W\frac{d^2 F}{dt^2}-3W^2\frac{dr}{dt}-3W\frac{dW}{dt}r+\frac{dW}{dt}\frac{dF}{dt}+W^3\frac{\partial^2 F}{\partial s^2} \tag{14}$$

Let us differentiate both sides of equation (3) with respect to s. Making use of the continuity equation (1) we get:

$$\frac{\partial^2 F}{\partial s^2}\left(U^2-\frac{gF}{\alpha B}\right)=-\frac{\partial^2 F}{\partial t^2}+\frac{2}{F}\left(\frac{\partial F}{\partial t}\right)^2+\frac{\partial F}{\partial s}\left\{\frac{6U}{F}\frac{\partial F}{\partial t}+\right.$$
$$\left.+\left(\frac{3U^2}{F}-\frac{gF}{\alpha B^2}\frac{dB}{\partial F}\right)\frac{\partial F}{\partial s}-\frac{1}{F}\frac{\partial Q}{\partial t}-\frac{g}{\alpha}\frac{U^2}{C^2 R}\left[2+M(1+\beta)\right]\right\}-$$
$$-\frac{g}{\alpha}\frac{2U}{C^2 R}\frac{\partial F}{\partial t}-2U\frac{\partial^2 F}{\partial s\,\partial t} \tag{15}$$

By means of (8), (9), (11) and (12) we obtain for r an ordinary differential equation of the first order in the general form

$$\frac{dr}{dt}=\frac{3}{4}\frac{W^2}{2U-W}\frac{N}{F}\left(\frac{dF}{ds}\right)^2+\frac{W}{4F}\left(9+\frac{F}{B}\frac{dB}{dF}\right)\frac{dF}{ds}r-Er-$$
$$-\frac{3}{2}\frac{W-U}{F}Nr^2-\frac{g}{\alpha}\frac{UW}{C^2 R(2U-W)}\left[1+\frac{M(1+\beta)U}{2(W-U)}\right]\frac{dF}{ds} \tag{16}$$

Here and henceforth

$$E=\frac{g}{\alpha}\frac{U}{C^2 R}\left[1-\frac{M(1+\beta)U}{2(W-U)}\right] \tag{17}$$

From equation (16) r can be computed for any position of the front, since the value of r_i at point ($s=0$, $t=0$) is determined, according to the condition (11), by the formula

$$r_i=\left(\frac{\partial F}{\partial s}\right)_i=\left(\frac{dF}{ds}\right)_i-\frac{1}{W_i^2}\left(\frac{\partial Q}{\partial t}\right)_i \tag{18}$$

We proceed to construct the differential equation for the derivatives $\partial^2 F/\partial s^2=\delta$. We shall have

$$\frac{d\delta}{dt}=\frac{\partial^3 F}{\partial s^3}W+\frac{\partial^3 F}{\partial s^2\,\partial t} \tag{19}$$

Taking the total derivative of $\partial^2 F/\partial s\,\partial t$ with respect to time t and making use of (19), we find

$$\frac{\partial^3 F}{\partial s\,\partial t^2}=\frac{d^2 r}{dt^2}-2W\frac{d\delta}{dt}-\delta\frac{dW}{dt}+W^2\frac{\partial^3 F}{\partial s^3} \tag{20}$$

Taking the third total derivative of F with respect to t, we can find the expression of $\partial^3 F/\partial t^3$ through $\partial^3 F/\partial s^3$ similar to (13), and from the condition that $d^3 Q/dt^3=0$ we can express $\partial^3 Q/\partial t^3$ through $\partial^3 F/\partial s^3$ as in (14).

Thus, one of the third derivatives remains arbitrary. When the value of the one of the third derivatives at the wave-front is given, the other two can be obtained by computation. Let $\partial^3 F/\partial s^3$ be the arbitrary derivative. Differentiating both sides of equation (15) with respect to s and making use of conditions (19) and (20), we shall obtain the ordinary differential equation of the first order for the derivative δ, which it was our object to find and which after substituting $\partial^2 F/\partial s^2 = \delta$ (conditions (12) and (13)) for $\partial^2 F/\partial t^2$ and $\partial^2 F/\partial s\, \partial t$, and $\partial F/\partial s = r$ (condition (11)) for $\partial Q/\partial t$ and $\partial F/\partial t$, will take the general form

$$d\delta = [\delta f_1(t) + f_2(t)]\, dt \tag{21}$$

Here the functions $f_1(t)$ and $f_2(t)$ involve W, U, F and $r = \partial F/\partial s$ which, as is obvious from the foregoing, can be represented as functions of s or t.

For the case where the uniform flow is disturbed by the wave, we get

$$\begin{aligned} \frac{d\delta}{dt} = &-\delta\left(\frac{g}{2}\frac{W-U}{F}Nr + E\right) - r\frac{E}{2(W-U)}\frac{g}{\alpha}\frac{U}{C^2R}\Bigg(1 + \\ &+\frac{M(1+\beta)U}{2(W-U)}\Bigg)r + r^2\frac{g}{\alpha C^2RF}\Bigg\{\left(\frac{3}{4}N-2\right)U - W + \\ &+\frac{UM(1+\beta)}{W-U}\left(2W - \frac{9}{8}NU\right) - \frac{U^2}{2(W-U)}\left[M^2(1+\beta)^2 - F(1+\beta)\frac{dM}{dF}\right]\Bigg\} + \\ &+ r^3\frac{W-U}{F^2}\left[\frac{3}{4} - \frac{F}{B}\frac{dB}{dF} - \frac{3}{4}\frac{F^2}{B^2}\left(\frac{dB}{dF}\right)^2 + \frac{1}{2}\frac{F^2}{B}\frac{d^2B}{dF^2}\right] \end{aligned} \tag{22}$$

The investigation of the expressions (15) and (22) will be reported elsewhere.

Equations to compute derivatives of higher orders can be obtained in a similar manner.

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